

Non-equilibrium glass transitions in driven and active matter

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The glass transition, extensively studied in dense fluids, polymers or colloids, corresponds to a marked evolution of equilibrium transport coefficients on a modest change of control parameter, such as temperature or pressure. A similar phenomenology is found in many systems evolving far from equilibrium, such as driven granular media, active and living matter. Although many theories compete to describe the glass transition at thermal equilibrium, very little is understood far from equilibrium. Here, we solve the dynamics of a specific, yet representative, class of glass models in the presence of non-thermal driving forces and energy dissipation, and show that a dynamic arrest can take place in these non-equilibrium conditions. Whereas the location of the transition depends on the specifics of the driving mechanisms, important features of the glassy dynamics are insensitive to details, suggesting that an effective thermal dynamics generically emerges at long timescales in non-equilibrium systems close to dynamic arrest.

Our goal is to study theoretically whether a glass transition can occur in a many-body interacting system driven by fluctuations that do not have a thermal origin. We motivate this very general question using recent experimental findings, which we divide into two broad classes. First, in the presence of mechanical forcing, granular media exhibit a form of dynamical slowing down that shares important similarities with the dynamics of molecular liquids near the glass transition. Forces driving the dynamics of the grains can, for instance, be a global oscillatory shear¹, high-frequency vertical vibrations^{2,3} or an air flow through the granular assembly⁴. The injected mechanical energy produces erratic grain motions, and energy is dissipated through collisions. Dense granular fluids driven through dynamic arrest are also studied numerically⁵. Active and living materials, such as self-propelled colloids and grains^{6–8}, cells and bacteria^{9,10} form a second class of non-equilibrium materials where glassy effects are reported and a dynamic arrest is observed at large density. Numerical studies of simple models of active matter in the dense regime have recently been published^{11–14}, and similarities to the equilibrium glass problem were noted⁹. More generally, the issue of mapping the far-from-equilibrium dynamics of active and self-propelled particles to an effective equilibrium problem is a recurring theoretical question^{14–17}, for which we are able to provide an explicit solution in the particular context of the dynamics of driven glassy materials.

Previous investigations either assume from the start, or demonstrate through detailed observations, that the observed marked change in dynamic properties is similar to the glass transition of simple fluids at thermal equilibrium¹⁸, in the sense that typical signatures of glassy dynamics (caging effect, two-step decay of time-correlation functions, dynamic heterogeneity and so on) are found. Theoretical studies exist for particular systems^{5,11,19}, and this broad set of experimental observations raises, we believe, a more general question that was not addressed in previous work. Can a glass transition occur far from thermal equilibrium, and how similar is it to the corresponding equilibrium phenomenon?

Here we address this question from a theoretical viewpoint. We study the effect of non-thermal sources of energy injection and dissipation on the slow dynamics of glassy materials. To obtain

quantitative results, we focus on a specific, yet representative, class of glass models whose equilibrium behaviour can be studied analytically and is well understood²⁰, following exactly the general scenario obtained in the context of random first-order transition theory²¹. Our main result is that glass transitions might exist even when driving forces are not thermal. Our results also indicate that the location of the transition continuously depends on the microscopic details of the dynamics, but that slow relaxation near dynamic arrest is insensitive to these details, thus suggesting that an effective equilibrium glassy dynamics generically emerges at long times in non-equilibrium materials close to dynamic arrest.

To model the experimental situations described above we consider the dynamics of N degrees of freedom, $\mathbf{x} = \{x_i, i = 1 \dots N\}$, representing for instance the position of grains or cells, interacting through the Hamiltonian $H[\mathbf{x}]$, which is supposed to exhibit a glass transition at thermal equilibrium. The driven and active materials we wish to study share two important characteristics. First, they dissipate energy through internal degrees of freedom at a finite rate. Second, energy is continuously supplied either by a global external forcing or by the particles themselves. To account for these effects, we study the following equation of motion:

$$\dot{x}_i(t) + \int_{-\infty}^t ds \gamma_d(t-s)x_i(s) + \frac{\partial H}{\partial x_i} + \eta_i(t) + f_i^a(t) = 0 \quad (1)$$

where we have included contributions from a (white noise) equilibrium thermal bath at temperature T represented by force $\eta_i(t)$, which satisfies the fluctuation–dissipation theorem (FDT), $\langle \eta_i(t)\eta_j(s) \rangle = 2T\delta(t-s)\delta_{ij}$. We have also included the contributions from non-equilibrium, coloured driving and dissipative mechanisms represented by the active force $f_i^a(t)$ and the dissipation kernel $\gamma_d(t)$, respectively. Equation (1) is a standard theoretical model for the dynamics of active colloids and molecular motors far from equilibrium^{22–24}. It also represents a minimal model to analyse the physics studied in numerical treatments of active^{11,12} and self-propelled particles¹³, where particles perform persistent random walks. It certainly misses some features of more complicated situations, such as complex alignment rules or particle anisotropy²⁵.

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A more intriguing relation, valid when the friction (but not the noise) is Markovian, is associated with a Langevin process²⁶ that mimics a quantum equilibrium bath, and reproduces some features of the true (operator-valued) quantum Langevin equation²⁷. The parameters ϵ_a and τ_a used here are the counterparts of \hbar and $\tau_{\text{quant}} \equiv (\hbar/kT)$. We shall use this analogy below.

As a first step we choose simple functional forms for the coloured noise and dissipation terms. We use a Gaussian random forcing with mean zero and variance $\langle f_i^a(t)f_j^a(s) \rangle = 2F_a(t-s)\delta_{ij}$, where $F_a(t) = (\epsilon_a/\tau_a)\exp(-t/\tau_a)$, where τ_a is the timescale of the slow forcing. For the dissipation we similarly choose $\gamma_d(t) = (\epsilon_d/\tau_d)\exp(-t/\tau_d)$, which defines the timescale τ_d . With these definitions, thermal equilibrium is recovered either when $\epsilon_d = \epsilon_a = 0$, or when the coloured forces and friction satisfy the equilibrium FDT (ref. 28), $F_a(t) = T\gamma_d(t)$. We expect similar results for different functional forms for these correlators, as long as they describe noises with finite correlation times.

To make the problem tractable, we perform a mean-field approximation of the glass Hamiltonian. Our goal is to have a well-understood equilibrium starting point to isolate the influence of the non-thermal forces. We specialize our study to the spherical p -spin model, as a well-known representative microscopic model where the random first-order transition scenario becomes exact²⁰. It is defined by

$$H = - \sum_{i_1, \dots, i_p} J_{i_1 \dots i_p} x_{i_1} \dots x_{i_p} \quad (2)$$

for continuous spins obeying the spherical constraint $\sum_i x_i^2 = N$ and where $J_{i_1 \dots i_p}$ represents a random coupling constant. In short, our strategy is to perform a mean-field approximation to the interactions in the equation of motion (1), while retaining realistic forms for the sources of injection and dissipation. As usual when dealing with glassy dynamics, our theoretical predictions strictly hold within the particular context of random first-order transition theory, but we expect them to have wider physical relevance (see ref. 18 for a broad theoretical overview). Another advantage of our approach is that it provides precise predictions that are then useful guides to computer simulations of more realistic models of active particles.

As the Hamiltonian (2) is fully connected, closed and exact equations of motion can be derived for the autocorrelation function $C(t, s) = \langle x_i(t)x_i(s) \rangle$, and for the autoresponse function $R(t, s) = \partial \langle x_i(t) \rangle / \partial \eta_i(s)$:

$$\begin{aligned} \frac{\partial C(t, s)}{\partial s} &= -\mu(t)C(t, s) + \int_{-\infty}^s dt' D(t, t')R(s, t') \\ &\quad + \int_{-\infty}^t dt' \Sigma(t, t')C(t', s) + 2TR(s, t) \\ \frac{\partial R(t, s)}{\partial s} &= -\mu(t)R(t, s) + \int_{-\infty}^t dt' \Sigma(t, t')R(t', s) + \delta(t-s) \\ \mu(t) &= T + \int_{-\infty}^t dt' [D(t, t')R(t, t') + \Sigma(t, t')C(t, t')] \end{aligned} \quad (3)$$

with the kernels $D(t, s) = (p/2)C^{p-1}(t, s) + F_a(t-s)$ and $\Sigma(t, s) = (p(p-1)/2)C^{p-2}(t, s)R(t, s) + (\partial \gamma_d(t-s)/\partial s)$. The last equation in equations (3) enforces the spherical constraint.

Technically, introducing coloured friction and noise breaks detailed balance and introduces new physical timescales (τ_d and τ_a) that compete and perturb the equilibrium dynamics of the system. This situation is reminiscent of the driven dynamics of the model studied in previous work²⁹, where non-Hamiltonian driving forces were introduced to model an applied shear flow. The

crucial difference is the form of the driving terms, whose typical timescales in ref. 29 were those of the dynamics itself, whereas here they relax with their own, fixed timescales τ_d and τ_a (see equation (3)). As a result, whereas the equilibrium glass transition was found to disappear in the presence of power dissipation of infinitesimally small amplitude²⁹, we show here that the glass transition may also survive the introduction of fluctuating forces, even of large amplitude, a situation that is closer to the effect of an oscillating field³⁰.

The dynamic equations (3) are coupled integro-differential equations for which no general analytical solution can be obtained. Our strategy was to explore the region in parameter space where the dynamics becomes stationary, such that a driven steady state is reached and the system is in a fluid phase with a finite relaxation time. We obtained numerical solutions of the dynamical equations, which motivated an analytic investigation of their asymptotic properties.

We have adapted the numerical scheme of ref. 29 to obtain a solution of equations (3) for a broad range of space parameters, using the following values for the timescales τ_d and τ_a : (1, 3, 10, 30, 100), in units of the bare friction coefficient, which was set to unity in equation (1). We varied ϵ_d and ϵ_a to fully explore the fluid part of the phase diagram. Numerically, we are limited by the stability of the numerical algorithm used to solve equations that do not have a formally causal form in the time differences²⁹. We illustrate our results in Fig. 1 with the solution obtained in two limiting cases where either only dissipation or active driving forces are introduced. We do not find any added complexity when both terms are simultaneously present with different amplitudes and timescales.

A direct analytical solution exists only in the trivial limit where $\tau_d, \tau_a \rightarrow 0$, because equation (1) reduces to a standard Langevin dynamics with white noise and memoryless friction. Equilibrium is then achieved at the rescaled temperature

$$\bar{T} = \frac{T + \epsilon_a}{1 + \epsilon_d} \quad (4)$$

Below, we argue that for finite correlation times of driving and dissipation forces, a (more complicated) relation to the equilibrium situation holds only in the limit of very large timescales, a result that is far more subtle than the mapping in equation (4). Equation (4) implies in particular that response and correlation functions satisfy the FDT, $\bar{T}R(t) = -(dC(t)/dt)$, and that equations (3) reduce to

$$\frac{dC(t)}{dt} + \bar{T}C(t) - \frac{p}{2\bar{T}} \int_0^t dt' C^{p-1}(t-t') \frac{dC(t')}{dt'} = 0 \quad (5)$$

which is mathematically equivalent²⁰ to the so-called F_{p-1} schematic model derived in the context of the mode-coupling theory (MCT) of the glass transition³¹. Its solution is known in great detail, and exhibits a dynamic singularity as the (rescaled) temperature is lowered towards the equilibrium value T_c^{eq} . Near the dynamic transition, the correlation function develops a two-step decay, with asymptotic time dependences that follow the behaviour described for discontinuous (or type B) transitions within MCT (ref. 31). An expression for the critical temperature and scaling laws is obtained by performing a detailed mathematical analysis³¹ of the equation derived by taking the long-time limit of equation (5),

$$\bar{T}C_s(t) + \frac{1}{\bar{T}} \int_0^t dt' D_s(t-t') \frac{dC_s(t')}{dt'} - \frac{1-q}{\bar{T}} D_s(t) = 0 \quad (6)$$

where $D_s(t) = (p/2)C_s^{p-1}(t)$.

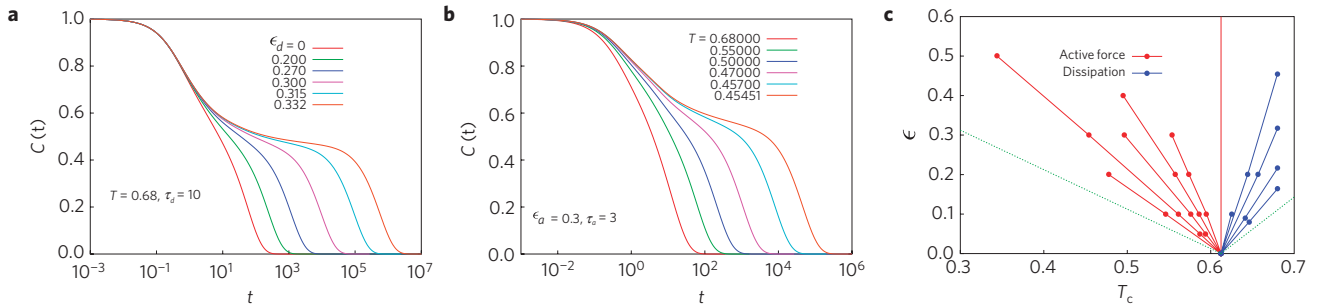


Figure 1 | Slow dynamics with coloured dissipation and noise. **a**, Time correlators obtained from the numerical solution of equation (3) for $p = 3$ at constant temperature: increasing the amplitude of the dissipation ϵ_d with $\tau_d = 10$ drives the system to the glass phase. **b**, Decreasing temperatures for constant driving force with $\epsilon_a = 0.3$, $\tau_a = 3$. **c**, Phase diagram of the system for active forces ($T_c < T_c^{\text{eq}}$) and slow dissipation ($T_c > T_c^{\text{eq}}$), for various timescales $\tau_d, \tau_a = 1, 3, 10, 30, 100$ (from bottom to top). The vertical line indicates T_c^{eq} ; dashed lines indicate the limits $\tau_d, \tau_a \rightarrow 0$, equation (4).

Note that, in the general driven case with a non-equilibrium bath, one can go from equations (3) to equation (5) only by formally eliminating $R(t-t')$ from the dynamic equations. In this way, one obtains a version of equation (5) with the memory kernel taking a form $\mathcal{M}[C](t-t')$, which is a complicated nonlocal functional of $C(t)$ (as opposed to the local functions such as $C^{p-1}(t)$).

Our central result is that the main features of this equilibrium glass transition robustly survive the introduction of a finite amount of non-thermal fluctuations driving the system far from thermal equilibrium, as illustrated in Fig. 1. In both cases of coloured noise or dissipation, we numerically find that time-correlation functions exhibit a two-step decay reminiscent of the equilibrium behaviour. This emerging glassy dynamics is characterized by a relaxation time that diverges on approaching a dynamic transition, which we call a non-equilibrium glass transition. We find that the location of the transition is a continuous function of the driving mechanism. The data in Fig. 1 confirm the natural expectation that T_c increases in the presence of the additional coloured dissipation, $T_c(\epsilon_d, \tau_d) > T_c^{\text{eq}}$, whereas it decreases in the presence of a coloured forcing, $T_c(\epsilon_a, \tau_a) < T_c^{\text{eq}}$. The phase diagram determined numerically in Fig. 1c suggests a nearly linear dependence of T_c on either ϵ_a or ϵ_d , with a less pronounced dependence on the timescales (note that τ_d and τ_a vary over two orders of magnitude in Fig. 1c).

We can analytically rationalize the above findings, and obtain further insight into the slow dynamics near non-equilibrium glass transitions, relying on the fact that the driving terms responsible for the explicit violation of detailed balance in the dynamical equations (3) have correlations that vanish at long times. Thus, a strong scale separation occurs when the structural relaxation is much larger than both τ_d and τ_a , that is, sufficiently close to the non-equilibrium transition. We seek an approximate equation of motion valid for stationary states in the limit of large times, $\tau_d, \tau_a \ll t$, corresponding to the approach and departure from the plateau:

$$\frac{\partial C_s(t)}{\partial t} = -(\mu - I_\Sigma)C_s(t) + I_R D_s(t) + \int_0^t dt' \Sigma_s(t-t')C_s(t') + \int_0^\infty dt' [D_s(t+t')R_s(t') + \Sigma_s(t+t')C_s(t')] \quad (7)$$

$$\frac{\partial R_s(t)}{\partial t} = -(\mu - I_\Sigma)R_s(t) + \Sigma_s(t)I_R + \int_0^t dt' \Sigma_s(t-t')R_s(t') \quad (8)$$

$$\mu = T + \Omega + \int_0^\infty dt' [D_s(t')R_s(t') + \Sigma_s(t')C_s(t')] \quad (9)$$

with $\Sigma_s = (p(p-1)/2)C_s^{p-2}R_s$, $D_s = (p/2)C_s^{p-1}$, and the following integrals $I_\Sigma = \int_0^\infty dt' \Sigma_f(t')$, $I_R = \int_0^\infty dt' R_f(t')$, and

$\Omega = \int_0^\infty dt' [D_f(t')R_f(t') + \Sigma_f(t')C_f(t')]$. We defined the slow functions $C_s(t)$ and $R_s(t)$ as the exact solutions of equations (7)–(9), and the fast ones are defined by difference, for example $C_f(t) = C(t) - C_s(t)$, and decay over timescales that do not diverge at the transition.

A crucial element of the dynamical equations (7)–(9) governing the long-time dynamics is that the terms responsible for the explicit breaking of detailed balance have disappeared. They appear very indirectly through time integrals over the short-time dynamical behaviour. As an immediate consequence, these equations can be considerably simplified because equations (7) and (8) reduce to the same equation if correlation and response satisfy

$$R_s(t) = -\frac{X_s}{T} \frac{dC_s(t)}{dt} \quad (10)$$

which defines the fluctuation–dissipation ratio, X_s , or equivalently an effective temperature $T_{\text{eff}} = T/X_s$ (ref. 32). A similar ansatz holds in the long-time limit of the ageing regime³³, that is, in the unperturbed glass phase, and in the equivalent limit of vanishing shear-like forces²⁹.

Combining equations (7), (9) and (10), we obtain

$$\left(T + \frac{p}{2}X_s q^p + \Omega - I_\Sigma - \frac{p}{2}q^{p-1}\right)C_s(t) + \frac{X_s}{T} \int_0^t dt' D_s(t-t') \frac{dC_s(t')}{dt'} - I_R D_s(t) = 0 \quad (11)$$

where q represents the intermediate plateau height of $C(t)$. This equation constitutes our main analytical achievement. It shows that, at sufficiently long times, the dynamical equation governing structural relaxation is equivalent to the one found for equilibrium relaxation, equations (5) and (6), showing that an equilibrium-like glassy dynamics emerges out of non-thermal forces driving the dynamics at short times. A similar conclusion was reached in ref. 5 for the specific case of a driven granular fluid, but here we find that this form holds only once the fast motion is averaged away. Even then, there remain, however, two important differences from the equilibrium case.

First, the coupling parameters determining the numerical value of the critical temperature are renormalized by the microscopic details of the driving forces through time integrals over the short-time dynamics, as can be seen by directly comparing equations (6) and (11). This explains our numerical finding that the location of the transition and the value of the plateau height continuously depend on the details of the microscopic dynamics (Fig. 1c). Thus, an analytic determination of the locus of the non-equilibrium glass transition requires solving not only equation (11),

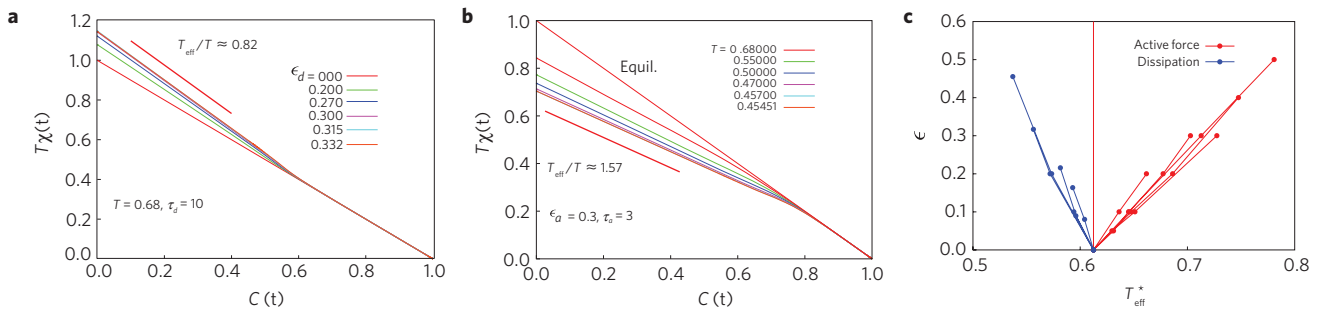


Figure 2 | Effective temperatures near non-equilibrium glass transitions. **a**, Dissipative forces produce $T_{\text{eff}} < T/X_s < T$ (that is, $X_s > 1$). Parameters as in Fig. 1a. **b**, Active driving produces $T_{\text{eff}} > T$ (that is, $X_s < 1$). Parameters as in Fig. 1b. **c**, Effective critical temperatures $T_{\text{eff}}^* = T_c/X_s$ for parameters as in Fig. 1c.

but also strongly non-universal features of the short-time dynamics. In contrast, because the long-time dynamics remains described by a discontinuous MCT transition, all universal features of the time-correlation functions remain valid far from equilibrium, as found numerically in Fig. 1. In particular, although the critical exponents describing the time dependence of correlation functions will depend on the driving forces, the well-known relations between them³¹ remain valid.

Second, whereas equation (11) simply involves the correlator $C_s(t)$, as in equilibrium, the response function $R_s(t)$ does not obey the equilibrium fluctuation–dissipation relation, but only an effective one, equation (10). Note that we predict the existence of a non-equilibrium T_{eff} for slow degrees of freedom even for the stationary fluid phase, not only deep into the glass as in ref. 34. This finding illustrates that non-equilibrium glass transitions are conceptually distinct from the equilibrium analogue, but that a form of equilibrium-like glassy dynamics naturally emerges at long times.

Our numerical analysis confirms the existence of effective temperatures (see Fig. 2). As usual^{32,33}, we represent integrated response functions, $\chi(t) = \int_0^t dt' R(t')$, versus $C(t)$ using t as a running parameter, which yields straight lines with slope $-X_s/T$ whenever equation (10) holds. The straight segments corresponding to the first step of the relaxation have no reason, despite appearances, to be straight because the system is strongly driven. The data in Fig. 2 show that X_s behaves differently if friction ($X_s(\epsilon_d, \tau_d) > 1$), or forcing ($X_s(\epsilon_a, \tau_a) < 1$), dominates the physics. The former represents an unusual situation where slow degrees of freedom seem to be colder than the bath³⁵. Although equation (4) cannot be used to predict the actual value of T_{eff} in the general case, it correctly predicts its qualitative trends and thus provides a simple physical argument for its variation with our control parameters. These two distinct cases are reminiscent of the distinction between adamant (hot) and susceptible (cold) molecular motors of ref. 19. Note also the striking similarity to the corresponding curves measured in active fluids^{11,12}.

Finally, we summarize our numerical findings for X_s measured at the dynamic transition into a phase diagram ($\epsilon, T_{\text{eff}}^* = T_c/X_s$) in Fig. 2c. With friction alone, we confirm that $T_{\text{eff}}^* < T_c(\epsilon_d, \tau_d)$ (that is, $X_s > 1$), but the resulting T_{eff}^* is lower than the equilibrium value T_c^{eq} . The additional dissipation thus drives the transition temperature T_c up, but it shifts the effective critical temperature T_{eff}^* down. Active forces have the opposite effect (see Fig. 2b,c). Disappointingly, the physical intuition¹⁹ that non-equilibrium transitions occur at shifted T_c but constant $T_{\text{eff}}^* \approx T_c^{\text{eq}}$ is not valid for our simple model: the effective temperature cannot be used to infer the non-equilibrium phase diagram in our model. This can also be seen analytically, as the mapping between equilibrium and off-equilibrium dynamics is more complicated than simply replacing T by T_{eff} , see equations (6) and (11).

Let us discuss the general picture of the (T, ϵ) phase diagram (Fig. 1c), in particular for the active forces. The red lines mark a limit between a liquid phase (higher T and ϵ) and a glass phase where the dynamics performs ageing. Our numerical method does not allow us to determine whether this transition line continues up to $T = 0$, as the numerical solution of the equations becomes unstable. One possibility is that the red lines continue and incide at the $T = 0$ axis without qualitative changes, thus enclosing the glassy phase. An alternative is suggested by the close analogy with the quantum Langevin equation described above. For the case of the (T, \hbar) plane, it is well established³⁶ that a continuous transition line (very much like our red lines) exists up to a point where the transition becomes thermodynamically first order all the way to $T = 0$ (see Fig. 2 of ref. 36). For our model, it would mean that the transition crossed by decreasing ϵ at constant (very) low temperature is first order: the system would vitrify suddenly from a liquid phase without any precursor increase in viscosity. Such an intriguing phenomenon, seemingly consistent with the data in ref. 13, clearly deserves further study.

Our study provides firm theoretical grounds to the emerging view that dynamic arrest in driven and active materials shares important similarities with the equilibrium glass transition, in particular regarding the behaviour of time-correlation functions and the emergence of an effective thermal behaviour. As our study was performed within a particular theoretical framework, it would be interesting to check the robustness of our findings beyond the realm of schematic glassy models, for instance using non-equilibrium mode-coupling types of approximations^{37,38}. Further away from mean-field approximations, one could for instance also test in actual measurements whether the role of activated processes¹⁸ remains the same in and out of equilibrium, and whether the mode-coupling crossover is as relevant in driven systems as it is in equilibrium ones.

Regarding effective temperatures, a numerical study of a kicked granular fluid⁵ reports a shift of the glass transition towards larger density with increasing activity, suggesting that drive dominates over friction in that case. Thus, we predict the existence of an effective temperature $T_{\text{eff}} > T$ for this system. Such a result is reported in numerical studies of active particle systems^{11,12}, but these measurements were not performed close to dynamic arrest. Closer to our ideas are the experimental measurements of an effective temperature in an air-fluidized granular bed³⁹, then used to infer a non-equilibrium equation of state. The reported deviations from equilibrium behaviour³⁹ are highly reminiscent of our findings in Fig. 2. We believe that future studies of active particles at large density¹³, or biologically driven systems²⁴, will result in more examples of non-equilibrium glass transitions as described in this work, where tests of our predictions about the universality of the slow dynamics and the emergence of effective temperatures could be performed.

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Author contributions

Both authors contributed equally to this work.

Additional information

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Competing financial interests

The authors declare no competing financial interests.